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EVAPORATION, HEAT TRANSFER, AND VELOCITY DISTRIBUTION IN TWO-DIMENSIONAL AND ROTATIONALLY SYMMETRICAL LAMINAR BOUNDARY-LAYER FLOW

By Nils Frössling

Translation of "Verdunstung, Wärmeübergang und Geschwindigkeitsverteilung bei zweidimensionaler und rotationssymmetrischer laminarer Grenzschichtströmung." Lunds Universitets Årsskrift, N.F. Avd. 2, Bd. 36, Nr. 4 Kungl. Fysiografiska Sällskapets Handlingar, N.F. Bd. 51, Nr. 4, 1940.

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EVAPORATION, HEAT TRANSFER, AND VELOCITY DISTRIBUTION IN TWO-DIMENSIONAL AND ROTATIONALLY SYMMETRICAL LAMINAR BOUNDARY-LAYER FLOW*

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INTRODUCTION

Aside from the simple case of the plane, no quantitative calculations of the evaporation of a body in a moving medium exist so far. The heat transfer, which under certain circumstances (see p. 4) follows the same laws, has been treated theoretically for the cylinder by Kroujiline (ref. 1) and Squire (ref. 2). For boundary-layer flow, Kroujiline used for the temperature field a power-series method of the same type which has been introduced for velocities by Pohlhausen (ref. 3). Because of this stipulation of the profile form, the result must be approximate, and the eventual agreement with the correct value is rather accidental. Squire gave an exact treatment of the transfer in the immediate proximity of the stagnation point. However, it is of great interest to have a calculation method for heat and mass transfer in the entire boundary layer, the error of which depends only on the work expenditure of the numerical calculation and, therefore, not on possible approximative formulations. Even though the calculation is time consuming, one has the advantage of being able to check approximate and more rapid methods with respect to this solution. Under the supposition that the constants of the problem (shape of body, pressure distribution, etc.) may possibly be eliminated from the equations to be solved, it is also possible in several cases to use the complete exact solution directly. The author of this report perfected, for this reason, the exact solutions for the temperature and concentration fields. Two dimensional and rotationally symmetrical steady boundary-layer flows were treated. The latter case is the more complicated one because of the form of the continuity equation.

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*"Verdunstung, Wärmeübergang und Geschwindigkeitsverteilung bei zweidimensionaler und rotationssymmetrischer laminarer Grenzschichtströmung." Lunds Universitets Årsskrift, N.F. Avd. 2, Bd. 36, Nr. 4 Kungl. Fysiografiska Sällskapets Handlingar, N.F. Bd. 51, Nr. 4, 1940.

For the calculation of the transfers, however, the velocity fields must be known. For the two-dimensional case one has, aside from approximate methods of solution by Pohlhausen (ref. 3), Kármán and Millikan (ref. 4), and others, the method of Blasius (ref. 5) and Hiemenz (ref. 6) which was improved by Howarth (ref. 7). By means of this method one may solve the equations, without any arbitrary assumptions regarding velocity profile and the like, by power-series development from the stagnation point up to an arbitrary point on the meridian curve. The work expenditure depends on the required accuracy and on the position of that point. Since the development becomes very rapidly more cumbersome with the distance from the stagnation point, it is appropriate to use, from a certain point onward, a continuation method, for instance, according to Prandtl (ref. 8) and Görtler (ref. 9). Howarth (ref. 7) transformed the functions of Blasius and Hiemenz into functions of such a type that the constants disappear from the equations so that the numerical solutions for them may be applied to any two-dimensional flow. He treated the symmetrical as well as the unsymmetrical case, and indicates the solution of one of these functions. For the present investigation, Howarth's functions are used in the two-dimensional case. Since the accuracy of Howarth's numerical tables is not sufficient for the calculation of the transfer, a new numerical calculation was made of certain functions. For the three-dimensional rotationally symmetrical case, in contrast, there exists, so far, no calculation of the functions of the series development. Because of the modified continuity equation, other systems of equations must be used, and these systems are established here. The necessary functions are numerically calculated. Although the form of the meridian curve takes effect through the continuity equation, one can proceed in the distribution of the functions in such a manner that the constants of the meridian equation disappear, and the solutions therefore are valid not only for arbitrary pressure distribution but also for arbitrary shape of the body of revolution. The continuation method, beginning with the limit of validity of the broken-off power series, has been perfected also for this case.

An investigation is carried out regarding the validity of the law, stated, for instance, by Ulsamer (ref. 10) that the Nusselt number is proportional to the cube root of the Prandtl number.

A few approximation methods for the calculation of the transfer layer are discussed.

Only a brief survey is presented here since a more detailed report is to be given in a later paper.

THE FUNDAMENTAL EQUATIONS

With dimensioned quantities, the boundary-layer equations for flow, concentration, and temperature read

$$\begin{cases} u \dfrac{\partial u}{\partial x} + v \dfrac{\partial u}{\partial y} = UU' + \nu \dfrac{\partial^2 u}{\partial y^2} \\[2ex] \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \quad \text{two dimensional} \\[2ex] \dfrac{\partial (ur)}{\partial x} + \dfrac{\partial (vr)}{\partial y} = 0 \ \text{rotationally symmetrical} \\[2ex] u \dfrac{\partial c}{\partial x} + v \dfrac{\partial c}{\partial y} = \Delta \dfrac{\partial^2 c}{\partial y^2} \\[2ex] u \dfrac{\partial t}{\partial x} + v \dfrac{\partial t}{\partial y} = a \dfrac{\partial^2 t}{\partial y^2} \end{cases}$$

Concerning the derivation of the two last equations for the rotationally symmetrical case see p. 15-16. The boundary conditions are

$$\begin{cases} y = 0; \quad u = v = 0; \quad t = t_0; \quad c = c_m; \\ y = \infty; \quad u = U; \quad t = 0; \quad c \doteq 0. \end{cases}$$

In these equations the customary designations of the various quantities are used. x = distance along the body surface from the stagnation point to the base point of the normal to the body surface. y = length of that normal. r = distance from base point to axis of rotation. u, v = velocity components in the direction of x and y, respectively. U = the velocity component parallel to the body surface immediately outside of the boundary layer (calculated from the experimentally determined pressure distribution). t = excess of the temperature of the surface over the temperature of the undisturbed fluid. c = corresponding concentration quantity. ν = kinematic viscosity, a = temperature diffusivity $\left(= \dfrac{\lambda}{\rho c_p}\right)$. Δ = diffusion coefficient. If

U_0 is the undisturbed velocity and D a characteristic length of the body (for instance, its diameter), one can transform the equation into dimensionless form by dividing the velocities, lengths, temperatures, and concentrations by the quantities U_0, D, t_0, and c_m. These equations contain as constants, among others, the Reynolds number $Re = \frac{UD}{\nu}$. This number one may eliminate by modifying the scale of the boundary layer in transverse direction, by multiplying the values of y and $\dot{v}$ by $\sqrt{Re}$. The equations used below with the designation "dimensionless equations without Reynolds numbers" are changed in their appearance, compared to those mentioned above, only in that ν disappears, and Δ and a are replaced by $\frac{\Delta}{\nu}$ and $\frac{a}{\nu}$. In the boundary conditions t_0 and c_m are replaced by 1. The two quantities $\frac{\Delta}{\nu}$ and $\frac{a}{\nu}$ which are often independent of pressure and temperature, as in the case of ideal gases, are dependent on the media used. These quantities are called Stanton's numbers. Frequently their inverse values are used, designated as Prandtl numbers. In an earlier report of the author on the evaporation of drops (ref. 11) the designation σ is used for the Stanton number. Since at present this letter is used mostly for the Prandtl numbers, this definition is employed in the present report to prevent misunderstandings. Thus σ here signifies: $\sigma = \frac{\nu}{\Delta}$ or, respectively, $= \frac{\nu}{a} = \frac{\nu c_p \rho}{\lambda}$.

The equations for temperature and concentration are therefore identical when t and c are interchanged. For the temperature-boundary layer it is assumed, however, that the dissipation and the heat generated by change in pressure may be neglected. This assumption is satisfied for not-too-large velocities (ref. 12). The equations also presuppose that the velocities be small compared to sonic velocity in order to make the compressibility negligible. A further limitation of the equations is given by the fact that the differences in concentration and temperature must not be so large that the constant characteristics of the media vary from point to point. Because of the identical form of the two equations for temperature and concentration, which is thus satisfied under these presuppositions, both may be treated simultaneously. In the following equations one may, therefore, immediately interchange the quantities c and t.

In the search for a solution which satisfies the accuracy requirements discussed in the introduction, the method of power-series development in x was used. Breaking off the power series after a certain number of terms one was able to use this solution from the stagnation point up to a point the position of which was dependent on the accuracy requirements. Starting from this point one could then use for the layers of different types step-by-step continuation methods.

For the sake of brevity, we shall use below the common name "transfer" boundary layer for the temperature and for the concentration-boundary layer.

POWER-SERIES DEVELOPMENTS IN x

A. TWO DIMENSIONAL CASE

a. Flow Boundary Layer

1. Symmetrical case

As was mentioned in the introduction, this case has been treated by Blasius, Hiemenz, and Howarth. This report uses for the most part the same designations as Howarth. The only difference is that Howarth's quantities F_ν are here replaced by the quantities ψ_ν because the capital letters are more suitable for the functions of the transfer boundary layer.

In order to replace the two unknown quantities u and v by a single one (ψ), the following conditions satisfying the continuity equation are set up as usual:

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x}$$

The first flow equation then becomes (with dimensionless quantities without Reynolds numbers)

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = UU' + \frac{\partial^3\psi}{\partial y^3}$$

Blasius and Hiemenz solved this equation by means of the formula

$$\psi = \psi_1 x + \psi_3 x^3 + \psi_5 x^5 + \ldots$$

for the symmetrical case where the velocity distribution outside of the boundary layer follows the formula $U = u_1 x + u_3 x^3 + u_5 x^5 + \ldots$. ψ_ν are functions only of y.

By comparison of the various powers of x the equations for ψ_ν were obtained. These equations were freed of the constants u_ν by introduction of the functions f_ν, g_ν, etc., by means of the following statements:

$$\eta = y\sqrt{u_1} \qquad \psi_1 = f_1\sqrt{u_1} \qquad \psi_3 = \frac{4u_3}{\sqrt{u_1}} f_3$$

$$\psi_5 = \frac{6u_5}{\sqrt{u_1}}\left(g_5 + \frac{u_3^2}{u_1 u_5} h_5\right) \qquad \psi_7 = \frac{8u_7}{\sqrt{u_1}}\left(g_7 + \frac{u_3 u_5}{u_1 u_7} h_7 + \frac{u_3^3}{u_1^2 u_7} k_7\right)$$

$$\psi_9 = \frac{10u_9}{\sqrt{u_1}}\left(g_9 + \frac{u_3 u_7}{u_1 u_9} h_9 + \frac{u_5^2}{u_1 u_9} k_9 + \frac{u_3^2 u_5}{u_1^2 u_9} j_9 + \frac{u_3^4}{u_1^3 u_9} q_9\right) \ldots$$

2. Unsymmetrical case

For an unsymmetrical two-dimensional body for which the velocity distribution follows the formula $U = u_1 x + u_2 x^2 + u_3 x^3 + \ldots$, the formulation

$$\psi = \psi_1 x + \psi_2 x^2 + \psi_3 x^3 + \ldots$$

was used. Here also the ψ_ν were freed of the constants u_ν, this time by the expressions

$$\eta = y\sqrt{u_1} \qquad \psi_1 = f_1\sqrt{u_1} \qquad \psi_2 = \frac{3u_2}{\sqrt{u_1}} f_2 \text{ etc.}$$

b. Transfer Boundary Layer

1. Symmetrical case

The author of this report attempted in the dimensionless equations without Reynolds numbers, aside from the formulations for ψ mentioned above, a development for c in the following manner (c_ν are functions of y only):

$$c = \sum_{\nu=0}^{\infty} c_\nu x^\nu = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots$$

Boundary conditions:

$$\begin{cases} y = 0 & c_0 = 1 \qquad c_1 = c_2 = \ldots = 0 \\ y = \infty & c_0 = c_1 = c_2 = \ldots = 0 \end{cases}$$

By substitution one obtains for the c_{2n} and c_{2n+1}

$$\left\{\begin{aligned} \frac{1}{\sigma} c''_{2n} &= \sum_{k=0}^{n} 2k\psi'_{2n+1-2k}c_{2k} - \sum_{0}^{n}(2n+1-2k)\psi_{2n+1-2k}c'_{2k} \\ \frac{1}{\sigma} c''_{2n+1} &= \sum_{k=0}^{n}(2k+1)\psi'_{2n+1-2k}c_{2k+1} - \sum_{0}^{n}(2n+1-2k)\psi_{2n+1-2k}c'_{2k+1} \end{aligned}\right\}$$

One can easily show that the equations for c_{2n+1} are such that they become identically zero. In the groups of equations mentioned above which constitute the recursion equations, there occur exclusively functions with even or odd subscripts. From this one can see that c is an even function of x which follows, besides, from the nature of the problem. In order to be free of the constants U_ν, new functions are introduced by the following statements:

$$\eta = y\sqrt{u_1}; \qquad \psi_\nu \text{ as above}; \qquad c_0 = F_0; \qquad c_2 = \frac{4u_3F_2}{u_1};$$

$$c_4 = \frac{6u_5}{u_1}\left(G_4 + \frac{u_3^2}{u_1u_5}H_4\right) \qquad c_6 = \frac{8u_7}{u_1}\left(G_6 + \frac{u_3u_5}{u_1u_7}H_6 + \frac{u_3^3}{u_1^2u_7}K_6\right)$$

$$c_8 = \frac{10u_9}{u_1}\left(G_8 + \frac{u_3u_7}{u_1u_9}H_8 + \frac{u_5^2}{u_1u_9}K_8 + \frac{u_3^2u_5}{u_1^2u_9}J_8 + \frac{u_3^4}{u_1^3u_9}Q_8\right)$$

Boundary conditions:

$\eta = 0$; $F_0 = 1$; the remaining functions = 0

$\eta = \infty$; all functions = 0

$$F_0 = 1 - \frac{\int_0^\eta e^{-\sigma\int_0^\eta f_1 d\eta} d\eta}{\int_0^\infty e^{-\sigma\int_0^\eta f_1 d\eta} d\eta}$$

The remaining equations do not have explicit solutions and must therefore be integrated by other methods, for instance, according to Runge and Kutta. (See p. 19.)

2. Unsymmetrical case

With $c = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$ and the boundary conditions

$$\begin{cases} y = 0; & c_0 = 1; \quad c_1 = c_2 = \ldots = 0 \\ y = \infty; & c_0 = c_1 = c_2 = \ldots = 0 \end{cases}$$

one obtains for c_ν

$$\frac{1}{\sigma} c_\nu'' = \sum_{k=0}^{\nu} k\psi'_{\nu+1-k} c_k - \sum_{0}^{\nu} (\nu + 1 - k)\psi_{\nu+1-k} c_k'$$

Here none of the functions c_ν disappears, and c is therefore, as had been expected, an even function. The equations here are not divided into two independent groups but the functions follow successively one from the other.

Distribution of the c_ν:

$$\eta = y\sqrt{u_1}; \quad \psi_1 = f_1\sqrt{u_1} \text{ etc.}; \quad c_0 = F_0; \quad c_1 = \frac{3u_2}{u_1} F_1;$$

$$c_2 = \frac{4u_3}{u_1}\left(G_2 + \frac{u_2^2}{u_1 u_3} H_2\right); \quad c_3 = \frac{5u_4}{u_1}\left(G_3 + \frac{u_2 u_3}{u_1 u_4} H_3 + \frac{u_2^3}{u_1^2 u_4} K_3\right);$$

$$c_4 = \frac{6u_5}{u_1}\left(G_4 + \frac{u_2 u_4}{u_1 u_5} H_4 + \frac{u_3^2}{u_1 u_5} K_4 + \frac{u_2^2 u_3}{u_1^2 u_5} J_4 + \frac{u_2^4}{u_1^3 u_5} Q_4\right);$$

Boundary conditions:

$$\begin{cases} \eta = 0; & F_0 = 1; \quad \text{the remaining functions} = 0; \\ \eta = \infty; & \text{all functions} = 0; \end{cases}$$

$$\frac{1}{\sigma}F_0'' = -f_1F_0'$$

$$\frac{1}{\sigma}F_1'' = -f_1F_1' + f_1'F_1 - 2f_2F_0'$$

$$\left.\begin{aligned} \frac{1}{\sigma}G_2'' &= -f_1G_2' + 2f_1'G_2 - 3g_3F_0' \\ \frac{1}{\sigma}H_2'' &= -f_1H_2' + 2f_1'H_2 - 3h_3F_0' + \frac{9}{4}\left(f_2'F_1 - 2f_2F_1'\right) \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{1}{\sigma}G_3'' &= -f_1G_3' + 3f_1'G_3 - 4g_4F_0' \\ \frac{1}{\sigma}H_3'' &= -f_1H_3' + 3f_1'H_3 - 4h_4F_0' + \frac{12}{5}\left(2f_2'G_2 + g_3'F_1 - 2f_2G_2' - 3g_3F_1'\right) \\ \frac{1}{\sigma}K_3'' &= -f_1K_3' + 3f_1'K_3 - 4k_4F_0' + \frac{12}{5}\left(2f_2'H_2 + h_3'F_1 - 2f_2H_2' - 3h_3F_1'\right) \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{1}{\sigma}G_4'' &= -f_1G_4' + 4f_1'G_4 - 5g_5F_0' \\ \frac{1}{\sigma}H_4'' &= -f_1H_4' + 4f_1'H_4 - 5h_5F_0' + \frac{5}{2}\left(3f_2'G_3 + g_4'F_1 - 2f_2G_3' - 4g_4F_1'\right) \\ \frac{1}{\sigma}K_4'' &= -f_1K_4' + 4f_1'K_4 - 5k_5F_0' + \frac{8}{3}\left(2g_3'G_2 - 3g_3G_2'\right) \\ \frac{1}{\sigma}J_4'' &= -f_1J_4' + 4f_1'J_4 - 5j_5F_0' + \frac{5}{2}\left(3f_2'H_3 + h_4'F_1 - 2f_2H_3' - 4h_4F_1'\right) + \\ &\quad \frac{16}{3}\left(g_3'H_2 + h_3'G_2\right) - 8\left(g_3H_2' + h_3G_2'\right) \\ \frac{1}{\sigma}Q_4'' &= -f_1Q_4' + 4f_1'Q_4 - 5q_5F_0' + \frac{5}{2}\left(3f_2'K_3 + k_4'F_1 - 2f_2K_3' - 4k_4F_1'\right) + \\ &\quad \frac{8}{3}\left(2h_3'H_2 - 3h_3H_2'\right) \end{aligned}\right\}$$

The first equation is identical with the first one of the symmetrical case.

B. ROTATIONALLY SYMMETRIC CASE

a. Flow Boundary Layer

For flows about a blunt body of revolution whose axis lies in the direction of the flow, the flow equations are, according to Boltze (ref. 13),

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = UU' + \nu\dfrac{\partial^2 u}{\partial y^2} \\[2ex] \dfrac{\partial(ur)}{\partial x} + \dfrac{\partial(vr)}{\partial y} = 0 \end{cases}$$

In the transformation to the dimensionless form without Reynolds number ν disappears. The quantity r then must have, for the bodies of revolution, the meaning, distance of axis of rotation up to the base point of the normal instead of up to the point (x, y). A function for identical solution of the continuity equation is desired. Boltze (ref. 13) suggested a function ψ which is defined as follows:

$$u = \frac{1}{r}\frac{\partial\psi}{\partial y} \qquad v = -\frac{1}{r}\frac{\partial\psi}{\partial x}$$

In the present report another solution $\bar{\psi}$ also has been examined. Definition

$$u = \frac{\partial\bar{\psi}}{\partial y} = \frac{1}{r}\frac{\partial(\bar{\psi}r)}{\partial y} \qquad v = -\frac{\partial\bar{\psi}}{\partial x} - \frac{\bar{\psi}}{r}\frac{dr}{dx} = -\frac{1}{r}\frac{\partial(\bar{\psi}r)}{\partial x}$$

The function $\bar{\psi}$ has the advantage that the equations become somewhat simpler and that the velocity profile is obtained directly. The functions are derived in both cases. ψ and $\bar{\psi}r$ are flow functions.

1. Use of the function ψ

After substitution of the expressions for u and v into the first boundary-layer equation one obtains

$$-\left(\frac{\partial\psi}{\partial y}\right)^2\frac{dr}{dx} + r\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - r\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = r^3UU' + r^2\frac{\partial^3\psi}{\partial y^3}$$

The power-series developments used (ψ_ν function of y; r_ν and u_ν constants)

$$\psi = \psi_2 x^2 + \psi_4 x^4 + \psi_6 x^6 + \dots$$
$$r = r_1 x + r_3 x^3 + r_5 x^5 + \dots$$
$$U = u_1 x + u_3 x^3 + u_5 x^5 + \dots$$

The functions ψ_ν have the following boundary conditions.

$$y = 0; \quad \psi_\nu = \psi_\nu' = 0;$$
$$y = \infty; \quad \psi_2' = r_1 u_1; \quad \psi_4' = r_1 u_3 + r_3 u_1; \quad \psi_6' = r_1 u_5 + r_3 u_3 + r_5 u_1; \dots$$

After substitution of the power expressions into the equation for ψ one obtains equations for ψ_ν by comparison of the different coefficients. These equations may be freed of the letters r_ν and u_ν by the following formulations:

$$\eta = y\sqrt{2u_1}; \quad \psi_2 = \frac{r_1 u_1}{\sqrt{2u_1}} f_2; \quad \psi_4 = \frac{2r_1 u_3}{\sqrt{2u_1}}\left(g_4 + \frac{r_3 u_1}{r_1 u_3} h_4\right)$$

$$\psi_6 = \frac{3r_1 u_5}{\sqrt{2u_1}}\left(g_6 + \frac{r_5 u_1}{r_1 u_5} h_6 + \frac{u_3^2}{u_1 u_5} k_6 + \frac{r_3 u_3}{r_1 u_5} j_6 + \frac{r_3^2 u_1}{r_1^2 u_5} q_6\right); \dots$$

The new functions have the boundary conditions

$$\eta = 0; \quad \text{all functions and their first derivative} = 0$$
$$\eta = \infty; \quad f_2' = 1; \quad g_4' = h_4' = \frac{1}{2}; \quad g_6' = h_6' = j_6' = \frac{1}{3}; \quad k_6' = q_6' = 0; \dots$$

One obtains the following equations:

$$f_2''' = -f_2f_2'' + \frac{1}{2}\left(f_2'^2 - 1\right)$$

$$\left.\begin{aligned}
g_4''' &= -f_2g_4'' + 2f_2'g_4' - 2f_2''g_4 - 1 \\
h_4''' &= -f_2h_4'' + 2f_2'h_4' - 2f_2''h_4 - \frac{1}{4}\left(3f_2'^2 - 2f_2f_2'' + 1\right)
\end{aligned}\right\}$$

$$\left.\begin{aligned}
g_6''' &= -f_2g_6'' + 3f_2'g_6' - 3f_2''g_6 - 1 \\
h_6''' &= -f_2h_6'' + 3f_2'h_6' - 3f_2''h_6 - \frac{1}{6}\left(5f_2'^2 - 2f_2f_2'' + 1\right) \\
k_6''' &= -f_2k_6'' + 3f_2'k_6' - 3f_2''k_6 + \frac{2}{3}\left(3g_4'^2 - 4g_4g_4''\right) - \frac{1}{2} \\
j_6''' &= -f_2j_6'' + 3f_2'j_6' - 3f_2''j_6 + 4g_4'h_4' - \frac{8}{3}\left(g_4h_4'' + h_4g_4''\right) + \\
&\quad \frac{2}{3}\left(f_2g_4'' - 4f_2'g_4' + 2f_2''g_4 - 1\right) \\
q_6''' &= -f_2q_6'' + 3f_2'q_6' - 3f_2''q_6 + \frac{2}{3}\left(3h_4'^2 - 4h_4h_4''\right) + \frac{2}{3}\left(f_2h_4'' - 4f_2'h_4'\right) + \\
&\quad 2f_2''h_4 + \frac{1}{6}\left(5f_2'^2 - 2f_2f_2''\right)
\end{aligned}\right\}$$

2. Use of the function $\bar{\psi}$

One obtains

$$\frac{\partial\bar{\psi}}{\partial y}\frac{\partial^2\bar{\psi}}{\partial x\partial y} - \frac{\partial\bar{\psi}}{\partial x}\frac{\partial^2\bar{\psi}}{\partial y^2} - \frac{1}{r}\frac{dr}{dx}\bar{\psi}\frac{\partial^2\bar{\psi}}{\partial y^2} = UU' + \frac{\partial^3\bar{\psi}}{\partial y^3}$$

Power series developments

$$\bar{\psi} = \bar{\psi}_1x + \bar{\psi}_3x^3 + \bar{\psi}_5x^5 + \dots$$

$$r = r_1x + r_3x^3 + r_5x^5 + \dots$$

$$U = u_1x + u_3x^3 + u_5x^5 + \dots$$

Boundary conditions

$$y = 0; \quad \bar{\psi}_\nu = \psi_\nu' = 0;$$

$$y = \infty; \quad \bar{\psi}_\nu' = u_\nu$$

The functions $\bar{\psi}_\nu$ are here divided up as follows:

$$\eta = y\sqrt{2u_1}; \quad \bar{\psi}_1 = \frac{u_1}{\sqrt{2u_1}} f_1; \quad \bar{\psi}_3 = \frac{2u_3}{\sqrt{2u_1}} f_3 = \frac{2u_3}{\sqrt{2u_1}}\left(g_3 + \frac{r_3 u_1}{r_1 u_3} h_3\right)$$

$$\bar{\psi}_5 = \frac{3u_5}{\sqrt{2u_1}} f_5 = \frac{3u_5}{\sqrt{2u_1}}\left(g_5 + \frac{r_5 u_1}{r_1 u_5} h_5 + \frac{u_3^2}{u_1 u_5} k_5 + \frac{r_3 u_3}{r_1 u_5} j_5 + \frac{r_3^2 u_1}{r_1^2 u_5} q_5\right)$$

Boundary conditions

$\eta = 0$; all functions and their first derivative = 0;

$$\eta = \infty; \quad f_1' = 1; \quad g_3' = \frac{1}{2}; \quad g_5' = \frac{1}{3}; \quad h_3' = h_5' = k_5' = j_5' = q_5' = 0;$$

$$f_1''' = -f_1 f_1'' + \frac{1}{2}\left(f_1'^2 - 1\right)$$

$$g_3''' = -f_1 g_3'' + 2f_1' g_3' - 2f_1'' g_3 - 1$$

$$h_3''' = -f_1 h_3'' + 2f_1' h_3' - 2f_1'' h_3 - \frac{1}{2} f_1 f_1''$$

$$g_5''' = -f_1 g_5'' + 3f_1' g_5' - 3f_1'' g_5 - 1$$

$$h_5''' = -f_1 h_5'' + 3f_1' h_5' - 3f_1'' h_5 - \frac{2}{3} f_1 f_1''$$

$$k_5''' = -f_1 k_5'' + 3f_1' k_5' - 3f_1'' k_5 + 2g_3'^2 - \frac{8}{3} g_3 g_3'' - \frac{1}{2}$$

$$j_5''' = -f_1 j_5'' + 3f_1' j_5' - 3f_1'' j_5 + 4g_3' h_3' - \frac{8}{3}\left(g_3 h_3'' + h_3 g_3''\right) - \frac{2}{3}\left(f_1'' g_3 + f_1 g_3''\right)$$

$$q_5''' = -f_1 q_5'' + 3f_1' q_5' - 3f_1'' q_5 + 2h_3'^2 - \frac{8}{3} h_3 h_3'' + \frac{1}{3} f_1 f_1'' - \frac{2}{3}\left(f_1 h_3'' + f_1'' h_3\right)$$

In the first method, the functions have even subscripts, in the second odd ones. A simple relation exists between the two groups of functions which one may easily obtain by equating the two expressions defining u and, respectively, the two expressions defining v (which gives $\psi = \bar{\psi} r$).

$$f_2 = f_1;$$

$$g_4 = g_3; \quad h_4 = h_3 + \frac{f_1}{2};$$

$$g_6 = g_5; \quad h_6 = h_5 + \frac{f_1}{3}; \quad k_6 = k_5; \quad j_6 = j_5 + \frac{2}{3} g_3; \quad q_6 = q_5 + \frac{2}{3} h_3$$

b. Transfer Boundary Layer

The general equation of the temperature and concentration fields for rotationally symmetrical flow has not been set up before. For the special case where the body is a sphere, the author (ref. 11) has shown that for boundary-layer flow the equation is identical with the one for the two-dimensional case, at least for points which do not lie directly at the stagnation point. In the present report, it is shown that the same boundary-layer equation is valid also for arbitrary blunt bodies of revolution, and that this applies to points directly at the stagnation point as well. The introduction of mass or heat into a volume element by diffusion and convection is expressed by the following equation (which is valid for rotationally symmetrical flow without neglect of the boundary layer when x and r are counted up to the element instead of to the base point):

$$\frac{\partial(cur)}{\partial x} + \frac{\partial(cvr)}{\partial y} = \Delta\left(\frac{\partial}{\partial x}\frac{\partial c}{\partial x} r\right) + \Delta \frac{\partial}{\partial y}\left(\frac{\partial c}{\partial y} r\right)$$

The derivation becomes the simplest if one chooses as the volume element an element bounded by two meridian planes, two surfaces x = constant, and two surfaces y = constant. In order to arrive at the boundary-layer equation, one groups the derivatives

$$\frac{c}{r}\frac{\partial(ur)}{\partial x} + \frac{c}{r}\frac{\partial(vr)}{\partial y} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = \Delta\left[\frac{\partial^2 c}{\partial y^2} + \frac{\partial c}{\partial y}\frac{1}{r}\frac{\partial r}{\partial y} + \frac{\partial^2 c}{\partial x^2} + \frac{1}{r}\frac{\partial r}{\partial x}\frac{\partial c}{\partial x}\right]$$

Supposing that a thin boundary layer exists, the terms 2 and 3 in the parenthesis disappear. The first two terms of the left side disappear because of the appearance of the continuity equation. The last term of the parenthesis becomes infinite at the stagnation point if $\frac{\partial c}{\partial x}$ is not here zero. In order to avoid discontinuities at the stagnation point, one must therefore equate there $\frac{\partial c}{\partial x} = 0$. Then the last term becomes everywhere negligible, and one obtains the equation

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = \Delta \frac{\partial^2 c}{\partial y^2}$$

which thus is identical with the one in the case of two-dimensional flow.

1. Use of the function ψ

Because of symmetry and of the requirement $\frac{\partial c}{\partial x} = 0$ one uses for c the expression $c = c_0 + c_2 x^2 + c_4 x^4 + \ldots$ with the following boundary conditions for c_ν:

$$y = 0; \quad c_0 = 1; \quad c_2 = c_4 = \ldots = 0;$$

$$y = \infty; \quad c_0 = c_2 = c_4 = \ldots = 0$$

For the c_ν one obtains equations which contain r_ν and u_ν. In order to eliminate these constants, one may make the following substitutions:

$$\eta = y\sqrt{2u_1}; \quad \psi_\nu \text{ as before;} \quad c_0 = F_0; \quad c_2 = \frac{2u_3}{u_1}\left(G_2 + \frac{r_3 u_1}{r_1 u_3} H_2\right)$$

$$c_4 = \frac{3u_5}{u_1} F_4 = \frac{3u_5}{u_1}\left(G_4 + \frac{r_5 u_1}{r_1 u_5} H_4 + \frac{u_3^2}{u_1 u_5} K_4 + \frac{r_3 u_3}{r_1 u_5} J_4 + \frac{r_3^2 u_1}{r_1^2 u_5} Q_4\right)$$

Boundary condition

$$\eta = 0; \; F_0 = 1; \text{ remaining functions} = 0;$$

$$\eta = \infty; \text{ all functions} = 0$$

For the new functions the following equations of the second order are obtained:

$$\frac{1}{\sigma} F_0'' = -f_2 F_0'$$

$$\frac{1}{\sigma} G_2'' = -f_2 G_2' + f_2' G_2 - 2g_4 F_0'$$

$$\frac{1}{\sigma} H_2'' = -f_2 H_2' + f_2' H_2 - 2h_4 F_0' + \frac{1}{2} f_2 F_0'$$

$$\frac{1}{\sigma} G_4'' = -f_2 G_4' + 2f_2' G_4 - 3g_6 F_0'$$

$$\frac{1}{\sigma} H_4'' = -f_2 H_4' + 2f_2' H_4 - 3h_6 F_0' + \frac{1}{3} f_2 F_0'$$

$$\frac{1}{\sigma} K_4'' = -f_2 K_4' + 2f_2' K_4 - 3k_6 F_0' + \frac{4}{3} g_4' G_2 - \frac{8}{3} g_4 G_2'$$

$$\frac{1}{\sigma} J_4'' = -f_2 J_4' + 2f_2' J_4 - 3j_6 F_0' + \frac{4}{3}\left(g_4' H_2 + h_4' G_2\right) - \frac{8}{3}\left(g_4 H_2' + h_4 G_2'\right) - \frac{2}{3}\left(- f_2 G_2' + f_2' G_2 - 2g_4 F_0'\right)$$

$$\frac{1}{\sigma} Q_4'' = -f_2 Q_4' + 2f_2' Q_4 - 3q_6 F_0' + \frac{4}{3} h_4' H_2 - \frac{8}{3} h_4 H_2' - \frac{2}{3}\left(-f_2 H_2' + f_2' H_2 - 2h_4 F_0' + \frac{1}{2} f_2 F_0'\right)$$

2. Use of the function $\bar{\psi}$

With the same power development for c and the same definitions of the functions F_0, G_2, G_4, etc., one obtains

$$\frac{1}{\sigma} F_0'' = -f_1 F_0'$$

$$\frac{1}{\sigma} G_2'' = -f_1 G_2' + f_1' G_2 - 2g_3 F_0'$$

$$\frac{1}{\sigma} H_2'' = -f_1 H_2' + f_1' H_2 - 2h_3 F_0' - \frac{1}{2} f_1 F_0'$$

$$\frac{1}{\sigma} G_4'' = -f_1 G_4' + 2f_1' G_4 - 3g_5 F_0'$$

$$\frac{1}{\sigma} H_4'' = -f_1 H_4' + 2f_1' H_4 - 3h_5 F_0' - \frac{2}{3} f_1 F_0'$$

$$\frac{1}{\sigma} K_4'' = -f_1 K_4' + 2f_1' K_4 - 3k_5 F_0' + \frac{4}{3} g_3' G_2 - \frac{8}{3} g_3 G_2'$$

$$\frac{1}{\sigma} J_4'' = -f_1 J_4' + 2f_1' J_4 - 3j_5 F_0' + \frac{4}{3}\left(g_3' H_2 + h_3' G_2\right) - \frac{2}{3}\left(f_1 G_2' + g_3 F_0'\right) - \frac{8}{3}\left(g_3 H_2' + h_3 G_2'\right)$$

$$\frac{1}{\sigma} Q_4'' = -f_1 Q_4' + 2f_1' Q_4 - 3q_5 F_0' + \frac{4}{3} h_3' H_2 - \frac{8}{3} h_3 H_2' - \frac{2}{3}\left(f_1 H_2' + h_3 F_0'\right) + \frac{1}{3} f_1 F_0'$$

One can show easily by application of the relationships between the functions with even and the functions with odd subscripts that the systems of equations for the cases 1 and 2 are identical.

The first equations of the two systems are identical with the first equation for two-dimensional flow and are, therefore, also solved by quadratures.

C. FINAL EXPRESSIONS FOR THE TRANSFER

The transfers are made dimensionless by the Nusselt number $Nu = \frac{D}{\Delta c_m} \frac{\partial^2 m}{\partial S \partial \tau}$ or, respectively, $\frac{D}{\lambda t_0} \frac{\partial^2 Q}{\partial S \partial \tau}$. It is easily shown that $\frac{Nu}{\sqrt{Re}} = \left(-\frac{\partial c}{\partial \eta}\right)_0 \frac{\partial \eta}{\partial y}$, where c and y are "dimensionless and without Reynolds number." The heat transferred by radiation is, of course, not contained in this expression. For two-dimensional symmetrical bodies one obtains

$$\frac{Nu}{\sqrt{Re}} = \left[-F_0'\sqrt{u_1} - \frac{4u_3F_2'}{\sqrt{u_1}} x^2 - \frac{6u_5}{\sqrt{u_1}}\left(G_4' + \frac{u_3^2}{u_1u_5} H_4'\right) x^4 - \ldots\right]_{\eta=0}$$

Corresponding expressions are obtained in the other cases. As one can see from the equations, one may easily calculate the Nusselt number for arbitrary pressure distributions and body shapes which agree with the formulations, if one has made a numerical calculation of the functions. Unfortunately, the quantity σ is left over and one must therefore make different solutions for different media. As is shown in a section below, however, one can free the equations of σ, too, if σ is large.

D. NUMERICAL CALCULATIONS

For the two-dimensional symmetrical case and for rotationally symmetrical bodies the author has numerically calculated various functions, corresponding to the three first terms of the power-series developments in x. The method of Runge and Kutta (ref. 14) was used for this purpose. This method is rather time consuming but one has good possibilities of determining the errors. The first function f_1 of the two-dimensional case has been calculated by Hiemenz and Howarth with an accuracy sufficient for this investigation; Howarth's values are directly used here. For f_1 in the rotationally symmetrical case there exists a table by Hartree (ref. 15) which was set up by using a mechanical differential analyzer. The accuracy is here not sufficient and the first two derivatives also are required; for this reason the function is calculated here anew. Since the equation for f_1 is not linear and one therefore cannot find the solution by combining two particular solutions, it was valuable to have approximate information on f_1'' for $\eta = 0$. The functions were solved mostly by steps of $\eta = 0.2$. Since the values with η-interval 0.1 must be known for the successive calculations, the values lying between were interpolated by means of a Taylor series. For the transfer boundary layer the calculations for the σ-value of the air (0.7) were performed in the two-dimensional symmetrical case because experimental results for the heat transfer of a circular cylinder in air exist (see, for instance, the compilation by Kroujiline (ref. 1). For $\left(F_0''\right)_0$ one may obtain values from a table given by Goldstein and calculated by Squire (ref. 2) also in the case of other σ-values. Squire indicated an analogous expression for the heat transfer at the stagnation point. For the rotationally symmetrical case calculations have been carried out for $\sigma = \frac{1}{0.395}$ because the only experimental result for the transfer distribution has been found for the evaporation of naphthalene spheres (ref. 11), naphthalene has this value of σ. If one wants to calculate the higher terms of the power-series development in x for a special case, one may combine the separate functions in a single term in order to save work expenditure; but the generality of the solution is lost thereby. This has been done for the boundary layer of the body of revolution

for $\sigma = \frac{1}{0.395}$. The parenthesis in the defining equation for G_4, H_4 etc., has been combined into a single function F_4 and calculated for the sphere for the pressure distribution of Fage. (See below.)

The tables of the calculated functions are printed at the end of the report with the exception of the higher ones for rotationally symmetric boundary layer. Here one has for $n = 0 \left(\text{and} \quad \sigma = \frac{1}{0.395}\right)$ $G_2' = 0.3186$, $H_2' = -0.1005$, and $F_4' = -0.2118$. The error of the tabulated functions which will be discussed in more detail later is at most a few units in the last digit.

From the tables one obtains for air, for the pressure distribution measured by Hiemenz (ref. 6) at $Re \sim 19000$ for the circular cylinder $U = 3.6314x - 2.1709x^3 - 1.5144x^5$

$$\frac{Nu}{\sqrt{Re}} = 0.9449 - 0.5100x^2 - 0.5956x^4 . . .$$

The quantity x is here dimensionless (the length dimension x divided by the diameter D). Not only in the range $0^\circ - 55^\circ$ where the series is to apply exactly (see E, 2) but up to the separation point this equation is in good agreement with the compilation of experimental distribution curves indicated by Kroujiline (ref. 1). The derivations will be discussed later.

For the sphere a qualitative agreement with the values obtained for evaporation of naphthalene at higher Re (ref. 11) is attained if the pressure distribution according to Fage (ref. 17) is used which gives $U = 3x - 3.4966x^3 + 4.7391x^5 - 5.4181x^7$ for $Re = 157200$ (ref. 18). One then obtains

$$\frac{Nu}{\sqrt{Re}} = 1.8615 - 2.1477x^2 + 2.4609x^4 . . .$$

The deviations depend, among other things, on the fact that Fage's pressure distribution is possibly not fulfilled for Reynolds numbers as small as those used here. Later on a more exact comparison will be made with more recent experimental values obtained by the author at still higher Reynolds numbers.

E. DETERMINATION OF THE ERRORS OF THE BROKEN-OFF POWER-SERIES DEVELOPMENTS

One may use various methods: (1) The following term is calculated and for not-too-large x this indicates the error. (2) If the value of the errors is not required with a very high accuracy, the coefficient of the x-terms may be assumed to be of the same order of magnitude. (3) Use of a continuation method of the profile. (See following section.) (4) In the case of direct differentiation, with respect to y, of the value of $\frac{\partial c}{\partial x}$ taken from the transfer boundary-layer equation one may obtain, for transition to $y = 0$, the first derivative of Nu in x whereby a continuation step may be taken directly with respect to Nu. Later on numerical calculations according to some of these methods will be given.

STEPWISE DEVELOPMENT OF THE BOUNDARY-LAYER PROFILE

A. TWO-DIMENSIONAL CASE

a. Flow Boundary Layer

Prandtl (ref. 8) indicated for this case a method which is based on the fact that one may obtain from the equations an expression for $\frac{\partial u}{\partial x}$ containing only u with derivatives for a prescribed x. $\frac{\partial u}{\partial x}$ becomes with dimensionless quantities without Reynolds numbers

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}\left[u \int_0^y \frac{1}{u^2}\left(UU' + \frac{\partial^2 u}{\partial y^2}\right) dy\right]$$

When two adjacent profiles (at $x - \Delta x$ and x) were known, for instance, by application of the method of Blasius and Hiemenz, it was possible to calculate a third for $x + \Delta x$. With the u-values at $x - \Delta x$ the $2\Delta x \frac{\partial u}{\partial x}$ values for x were used. One could then continue in the same manner with the profiles at x and $x + \Delta x$. In order to guarantee the convergence of the expression, one was not to use the original numerical profile at x but had to replace it by another which satisfied certain requirements. In order to calculate those, u was developed

into a power series with respect to y: $u = \sum_{\nu=1}^{\infty} \frac{a_\nu y^\nu}{\nu!}$. For the a_ν one obtained certain conditions by substitution into the flow equation whereby only some of them could be chosen arbitrarily. These latter were determined by comparison with the given profile. Görtler (ref. 9) perfected the method practically and used it in Hiemenz' pressure distribution over the circular cylinder. In the present report corresponding ideas are used for bodies of revolution and for the transfer boundary layer, and the necessary expressions are added and discussed.

b. Transfer Boundary Layer

From the basic equation one obtains directly

$$\frac{\partial c}{\partial x} = \frac{1}{u}\left(\frac{1}{\sigma}\frac{\partial^2 c}{\partial y^2} - v\,\frac{\partial c}{\partial y}\right)$$

This equation may therefore be used directly for step-by-step continuation of the vapor and temperature boundary layer. Conditions become here simpler insofar as no integration is necessary. However, here also the danger exists that the expression becomes uncertain at the wall (because of u occurring in the denominator). Moreover, $\frac{\partial c}{\partial x}$ must become identically zero at the wall. In order to satisfy the requirements, one resolves here also the quantity c into a power-series development with respect to $y\big(b_\nu$ function of x only$\big)$:

$$c = 1 - \sum_{\nu=1}^{\infty} \frac{b_\nu y^\nu}{\nu!}$$

By substitution one obtains

$$\sum \frac{a_\nu y^\nu}{\nu!} \sum \frac{b_\nu{}' y^\nu}{\nu!} - \sum \frac{a_\nu{}' y^{\nu+1}}{(\nu+1)!} \sum \frac{b_\nu y^{\nu-1}}{(\nu-1)!} = \frac{1}{\sigma} \sum \frac{b_\nu y^{\nu-2}}{(\nu-2)!}$$

By comparison of terms of the same degree, one arrives at the relation between the b_ν. For the first nine b_ν there applies (with $f = -UU'$)

b_1 free; $b_2 = b_3 = 0$

$\frac{b_4}{\sigma} = 2a_1b_1' - a_1'b_1$ free; $\frac{b_5}{\sigma} = 3fb_1' - f'b_1$; $b_6 = 0$

$$\frac{b_7}{\sigma} = 10a_1{}^2\sigma b_1{}'' + 5a_1a_1{}'(1 - 3\sigma)b_1{}' + \left[a_1{}'^2(10\sigma - 1) - a_1a_1{}''(5\sigma + 1)\right]b_1 \text{ free}$$

$$\frac{b_8}{\sigma} = 48a_1\sigma fb_1{}'' + 2\left[2a_1f'(3 - 7\sigma) - 15a_1{}'f\sigma\right]b_1{}' + \left[-2a_1f''(3\sigma + 1) - 15fa_1{}''\sigma + f'a_1 1(35\sigma - 2)\right]b_1;$$

$$\frac{b_9}{\sigma} = 63f^2\sigma b_1{}'' + 7ff'(2 - 9\sigma)b_1{}' + \left[f'^2(35\sigma - 2) - ff''(21\sigma + 2)\right]b_1$$

The free coefficients are calculated as before by comparison with the given profile, and the c-values developed in power series are substituted into the above equation for $\frac{\partial c}{\partial x}$.

B. ROTATIONALLY SYMMETRICAL CASE

a. Flow Boundary Layer

The equations read

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = UU' + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial(ur)}{\partial x} + \frac{\partial(vr)}{\partial y} = 0$$

Here r signifies, as before, for a blunt body of revolution the distance between the axis of rotation and the base point of the normal to the surface.

By eliminating $\frac{\partial u}{\partial x}$ one obtains a linear equation of the first order in v with solution

$$v = -u\int_0^y\left(\frac{UU'}{u^2} + \frac{1}{u^2}\frac{\partial^2 u}{\partial y^2}\right)dy - \frac{1}{r}\frac{dr}{dx}uy$$

By forming the derivative from v with respect to y and using the continuity equation one obtains

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}\left[u \int_{0}^{u} \frac{1}{u^2}\left(UU' + \frac{\partial^2 u}{\partial y^2}\right) dy\right] + \frac{1}{r}\frac{dr}{dx}\, y\, \frac{\partial u}{\partial y}$$

For given u-profile, one may therefore use an equation for the continuation of the boundary-layer profile which differs from the equation for the two-dimensional case only with respect to the last term. In order to establish the convergence at the wall, here also a power-series development in y becomes necessary

$$u = \sum_{1}^{\infty} \frac{a_\nu y^\nu}{\nu!}$$

since

$$v = -\sum_{1}^{\infty}\left(a_\nu' + a_\nu \frac{r'}{r}\right)\frac{y^{\nu+1}}{(\nu+1)!}$$

By substitution into the basic equations one obtains $\left(\text{with } f = -UU';\ g = \frac{r'}{r}\right)$

a_1 free; $a_2 = f$; $a_3 = 0$;

$a_4 = a_1 a_1' - a_1^2 g$ free; $a_5 = 2a_1 f' - 4a_1 fg$; $a_6 = 2ff' - 4f^2 g$

. . . .

As before, one determines the free coefficients.

b. Transfer Boundary Layer

With the same expression as above for the two-dimensional case one obtains for the b_ν

b_1 free; $b_2 = b_3 = 0$;

$\frac{b_4}{\sigma} = 2a_1b_1' - a_1'b_1 - a_1b_1g$ free; $\frac{b_5}{\sigma} = 3fb_1' - f'b_1 - fb_1g$; $b_6 = 0$;

. . . .

The practical execution in the last three cases will be discussed in a later report together with numerical calculations. The methods of continuation discussed yield results the accuracy of which depends exclusively on the work expenditure and is therefore not limited by postulating approximation functions. The methods may also be used for determining the accuracy of the aforementioned power-series developments in x in the case of breaking-off after a certain number of terms at a certain point. One then starts the continuation method at an x so small that the error is certainly small, and compares the result then obtained at a larger x with the one directly calculated from the power-series development in x.

DEPENDENCE OF THE EVAPORATION AND THE HEAT TRANSFER ON σ

A. GENERALITIES

Pohlhausen (ref. 16) has shown for the plane that Nu is approximately proportional to the quantity $\sqrt[3]{\sigma}$. In the approximate calculations of Kroujiline (ref. 1) the same was shown for the circular cylinder. Ulsamer (ref. 10) demonstrated that the law may be approximately selected from various experimental investigations on the heat transfer of a circular cylinder. The author of this report has confirmed the law at least approximately in the case of evaporation of drops (ref. 11).

From the equations of the section on power-series developments in x one sees that σ can probably not be eliminated from them by simple transformations. Thus one cannot expect a relation as simple as the aforementioned to apply exactly. For the case where σ is very large and the transfer boundary layer therefore thin compared to the flow boundary layer, the author of this report found the $\sqrt[3]{\sigma}$-law to be exact. In this case the curvature of the velocity profile may be neglected in the entire transfer boundary layer, and one may replace u by $(u')_0 y$ and v by $\frac{1}{2}(v'')_0 y^2$ in the general boundary-layer equation, with the apostrophes indicating derivatives with respect to y.

$$(u')_0 y \frac{\partial c}{\partial x} + \frac{(v'')_0}{2} y^2 \frac{\partial c}{\partial y} = \frac{1}{\sigma} \frac{\partial^2 c}{\partial y^2}$$

The variable $\zeta = y \sqrt[3]{\sigma}$ is introduced for c (not for u and v)

$$(u')_0 \zeta \frac{\partial c}{\partial x} + \frac{(v'')_0}{2} \zeta^2 \frac{\partial c}{\partial \zeta} = \frac{\partial^2 c}{\partial \zeta^2}$$

Boundary conditions: $\zeta = 0$; $c = 1$. $\zeta = \infty$; $c = 0$.

Thus one has obtained an equation free of σ. For this reason, c becomes $c = f\left(x, y \sqrt[3]{\sigma}\right)$; hence follows that for large σ the quantity Nu is proportional to the quantity $\sqrt[3]{\sigma}$ on the entire surface in the boundary layer. That the same law has been found experimentally also for a σ that is not large, is based on the fact that the quantity $Nu / \sqrt[3]{\sigma}$ does not vary greatly with σ and may therefore be found to be approximately constant in a small region.

B. TWO-DIMENSIONAL SYMMETRICAL CASE

Into the equations for F_0, F_2, G_4, H_4 . . . the following functions and variables are introduced:

$$\xi = \eta \sqrt[3]{\frac{\sigma}{6}\left(f_1''\right)_0}; \quad F_0(\eta) = \Phi_0(\xi); \quad F_2(\eta) = \frac{\left(f_3''\right)_0}{\left(f_1''\right)_0} \Phi_2(\xi)$$

$$G_4 = \frac{\left(g_5''\right)_0}{\left(f_1''\right)_0} \Gamma_4(\xi); \quad H_4 = \frac{\left(h_5''\right)_0}{\left(f_1''\right)_0} \Theta_4(\xi); \text{ etc}$$

Boundary conditions:

$\xi = 0$; $\Phi_0 = 1$; remaining functions $= 0$;
$\xi = \infty$; all functions $= 0$.

Taking into account that for large σ the equation $\psi = \frac{1}{2}(\psi'')_0 y^2$ is valid, one obtains

$$\Phi_0'' = -3\xi^2\Phi_0'$$

$$\Phi_2'' = -3\xi^2\Phi_2' + 12\xi\Phi_2 - 9\xi^2\Phi_0'$$

$$\Gamma_4'' = -3\xi^2\Gamma_4' + 24\xi\Gamma_4 - 15\xi^2\Phi_0'$$

$$\theta_4'' = -3\xi^2\theta_4' + 24\xi\theta_4 - 15\xi^2\Phi_0' + \frac{8\left(f_3''\right)_0^2}{\left(f_1''\right)_0\left(h_5''\right)_0}\left[4\xi\Phi_2 - 3\xi^2\Phi_2'\right]$$

. . . .

For the pressure distribution according to Hiemenz (see p. 20) one obtains in the case of a circular cylinder $\frac{Nu}{\sqrt[3]{\sigma}\sqrt{Re}} = 1.2592 - 0.7583x^2 \ldots$; for the case calculated above $\sigma = 0.7$ one obtains for the corresponding quantity $1.0642 - 0.5744x^2 - 0.6708x^4 \ldots$. Here x signifies the dimensionless abscissa which is obtained from the length dimension through division by the diameter D. The functions Φ_0 and Φ_2 are given numerically in table 6.

C. ROTATIONALLY SYMMETRICAL CASE

The following functions are now introduced:

$$\xi = \eta\sqrt[3]{\frac{\sigma}{6}\left(f_1''\right)_0};\quad F_0(\eta) = \Phi_0(\xi);\quad G_2(\eta) = \frac{\left(g_3''\right)_0}{\left(f_1''\right)_0}\Gamma_2(\xi);\quad H_2 = \frac{\left(h_3''\right)_0}{\left(f_1''\right)_0}\theta_2(\xi)$$

The equations become

$$\Phi_0'' = -3\xi^2\Phi_0'$$

$$\Gamma_2'' = -3\xi^2\Gamma_2' + 6\xi\Gamma_2 - 6\xi^2\Phi_0'$$

$$\theta_2'' = -3\xi^2\theta_2' + 6\xi\theta_2 - 6\xi^2\Phi_0' - \frac{3}{2}\frac{\left(f_1''\right)_0}{\left(h_3''\right)_0}\xi^2\Phi_0'$$

. . . .

As in the previous case, the solution of the first equation is

$$\Phi_0 = 1 - \frac{\int_0^{\xi} e^{-x^3}dx}{\int_0^{\infty} e^{-x^3}dx}$$

For the pressure distribution according to Fage (ref. 17) (see p. 20) one obtains for the sphere $\frac{Nu}{\sqrt[3]{\sigma}\sqrt{Re}} = 1.4723 - (. . .)x^2 + . . .$; for $\sigma = \frac{1}{0.395}$, one obtains 1.3658 - . . . .

DISCUSSION OF APPROXIMATE METHODS

As has been mentioned above, Pohlhausen (ref. 3) gave an approximate method for the solution of the boundary-layer equation for the flow about a circular cylinder. Tomotika (ref. 18) applied this method to the sphere. Kroujouline used a similar method for the transfer for the cylinder, applying a broken-off series development in y which was determined with utilization of the integral condition of the transfer boundary layer. For the flow boundary layer he used a parabolic profile whereby the agreement may be assumed to be bad particularly in the case of pressure increase. Probably better approximations could have been obtained with the use of polynominals of the fourth degree. These statements are valid only when the transfer boundary layer is thinner than the flow boundary layer. Here a brief description is given concerning some considerations of the author of this report concerning a body of revolution, for various relative magnitudes of the two layers.

The integral condition formerly not set up for bodies of revolution becomes (see p. 4)

$$\frac{1}{r}\frac{d}{dx}\left[r\int_0^{\delta} ucdy\right] = -\Delta\left(\frac{\partial c}{\partial y}\right)_0$$

which may be derived, for instance, by integration of the original equation. Here δ is the thickness of the transition boundary layer. Using dimensionless quantities without Reynolds numbers only, Δ is replaced by $\frac{1}{\sigma}$. This is assumed below.

If for the two profiles the definitions

$$\frac{u}{U} = a_1 \frac{y}{\delta_1} + a_2\left(\frac{y}{\delta_1}\right)^2 + a_3\left(\frac{y}{\delta_1}\right)^3 + a_4\left(\frac{y}{\delta_1}\right)^4$$

and

$$c = 1 - 2\frac{y}{\delta} + 2\left(\frac{y}{\delta}\right)^3 - \left(\frac{y}{\delta}\right)^4$$

are used, one obtains for $\delta < \delta_1$ the equation

$$\frac{2r}{\delta_1 z\sigma} = \frac{d}{dx}\left[rU\delta_1\left(\frac{a_1 z^2}{15} + \frac{a_2 z^3}{42} + \frac{3a_3 z^4}{280} + \frac{a_4 z^5}{180}\right)\right]$$

Here δ_1 is the thickness of the flow boundary layer and $z = \dfrac{\delta}{\delta_1}$.

For $\delta > \delta_1$, the integration is performed, with use of the integral condition, first from 0 to δ_1 and then from δ_1 to δ. Result

$$\frac{2r}{\delta_1 z\sigma} = \frac{d}{dx}\left\{rU\delta_1\left[\frac{3z}{10} + \left(\frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} + \frac{a_4}{5} - 1\right) - \frac{2}{z}\left(\frac{a_1}{3} + \frac{a_2}{4} + \frac{a_3}{5} + \frac{a_4}{6} - \frac{1}{2}\right) + \frac{2}{z^3}\left(\frac{a_1}{5} + \frac{a_2}{6} + \frac{a_3}{7} + \frac{a_4}{8} - \frac{1}{4}\right) - \frac{1}{z^4}\left(\frac{a_1}{6} + \frac{a_2}{7} + \frac{a_3}{8} + \frac{a_4}{9} - \frac{1}{5}\right)\right]\right\}$$

The two equations have the same form when the parentheses after δ_1 are denoted, for instance, by the letter P. From the first of the two equations one sees that, for a σ so large and a z therefore so small that only the first term of the parenthesis must be considered, this z is, for a given x, inversely proportional to the quantity $\sqrt[3]{\sigma}$. Since the Nusselt number Nu equals $\frac{2}{\delta_1 z}\sqrt{Re}$, Nu is, for a large σ, proportional to $\sqrt[3]{\sigma}$ also according to this approximate theory.

Since r, U, a_ν, and δ_1 are known functions of x, we have in any case an equation of the first order with z and x which can be solved with customary methods (for instance, with the isocline method

or according to Runge and Kutta). The only boundary condition required for this is the z-value at $x = 0$. This value is calculated from the equation $\lambda z \sigma P = 1$ where $\lambda = U'\delta_1^2$ is identical with the parameter λ used by Pohlhausen and Tomotika. For the sphere where $U' = 3$ and $\lambda = 4.716$, one obtains in the proximity of the stagnation point

$$\frac{1}{\sigma} = 0.8759z^3 - 0.2648z^4 + 0.01809z^5 + 0.00561z^6 \quad \text{or, respectively,}$$

$$\frac{1}{\sigma} = 1.4148z^2 - 1.2295z + 0.5052 - \frac{0.0746}{z^2} + \frac{0.0189}{z^3}$$

For a given z and therefore also given Nu one may easily calculate the corresponding σ. For

$z = 0.0 \quad 0.1 \quad 0.4 \quad 0.7 \quad 1.1 \quad 1.6 \quad 2.0 \quad 3.0 \quad 4.0$, one obtains

$Nu/\sqrt{Re}\,\sqrt[3]{\sigma} = 1.526 \quad 1.511 \quad 1.464 \quad 1.418 \quad 1.356 \quad 1.284 \quad 1.232 \quad 1.128 \quad 1.049$ and

$1/\sqrt[3]{\sigma} = 0 \quad 0.095 \quad 0.367 \quad 0.622 \quad 0.935 \quad 1.288 \quad 1.545 \quad 2.121 \quad 2.631$

The quantity $Nu/\sqrt{Re}\,\sqrt[3]{\sigma}$ is therefore, for a large σ, almost constant and varies in the proximity of the stagnation point about linearly with $1/\sqrt[3]{\sigma}$.

The reason for choosing, above, z instead of δ as the dependent variable was that z probably varies little with x (compare Kroujouline (ref. 1)) and can therefore be calculated exactly more easily.

In the later more detailed report on the investigations, the numerical results of this formulation as well as of others will be discussed. It was shown that the choice of the profile form had a great effect on the result.

SUMMARY

A preliminary report is given of a theoretical investigation of the boundary-layer flow for two-dimensional and rotationally symmetrical bodies. The evaporation, the heat transfer, and the velocity are calculated by power-series developments with respect to the meridian length.

The coefficient functions which were calculated numerically in some cases have been chosen so that the calculation is valid for all pressure distributions and body shapes. The methods for determination of the errors in breaking off the series are briefly treated. Methods of continuation are discussed. It is shown, for large Prandtl numbers, that the Nusselt number is exactly proportional to the cube root of the Prandtl number. Finally, approximate methods of calculation are discussed.

EPILOGUE

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TABLE 1. TWO-DIMENSIONAL SYMMETRICAL

FLOW BOUNDARY LAYER

η	f_3	f_3'	f_3''	g_5	g_5'	g_5''	h_5	h_5'	h_5''
0.0	0	0	0.7244	0	0	0.6348	0	0	0.1192
.1	.0035	.0675	.6249	.0030			.0005		
.2	.0132	.1251	.5286	.0114	.1072	.4402	.0017	.0141	.0249
.3	.0282	.1734	.4375	.0242			.0032		
.4	.0476	.2129	.3539	.0405	.1778	.2717	.0045	.0117	-.0436
.5	.0705	.2444	.2780	.0595			.0055		
.6	.0962	.2688	.2112	.0806	.2184	.1408	.0057	-.0010	-.0783
.7	.1240	.2869	.1530	.1030			.0052		
.8	.1534	.2997	.1037	.1264	.2367	.0483	.0039	-.0176	-.0833
.9	.1838	.3080	.0626	.1502			.0017		
1.0	.2149	.3125	.0292	.1742	.2399	-.0106	-.0012	-.0330	-.0680
1.1	.2462	.3140	.0028	.1981			-.0049		
1.2	.2776	.3132	-.0173	.2218	.2342	-.0431	-.0090	-.0441	-.0423
1.3	.3088	.3107	-.0320	.2449			-.0136		
1.4	.3397	.3070	-.0420	.2676	.2239	-.0567	-.0185	-.0498	-.0149
1.5	.3702	.3025	-.0482	.2897			-.0236		
1.6	.4002	.2974	-.0513	.3112	.2123	-.0580	-.0286	-.0503	.0088
1.7	.4297	.2923	-.0518	.3322			-.0336		
1.8	.4587	.2871	-.0506	.3526	.2012	-.0522	-.0384	-.0468	.0256
1.9	.4871	.2822	-.0480	.3724			-.0430		
2.0	.5151	.2775	-.0444	.3918	.1916	-.0432	-.0472	-.0406	.0351
2.1	.5426	.2733	-.0402	.4108			-.0510		
2.2	.5698	.2695	-.0358	.4293	.1839	-.0335	-.0546	-.0331	.0380
2.3	.5966	.2662	-.0314	.4476			-.0577		
2.4	.6230	.2632	-.0271	.4655	.1781	-.0245	-.0604	-.0257	.0361
2.5	.6492	.2607	-.0230	.4832			-.0628		
2.6	.6752	.2586	-.0194	.5007	.1740	-.0171	-.0649	-.0189	.0312
2.7	.7010	.2568	-.0160	.5180			-.0666		
2.8	.7266	.2554	-.0131	.5352	.1712	-.0114	-.0681	-.0133	.0249
2.9	.7520	.2542	-.0106	.5522			-.0693		
3.0	.7774	.2533	-.0085	.5692	.1694	-.0072	-.0703	-.0089	.0187
3.1	.8027	.2525	-.0067	.5861			-.0711		
3.2	.8279	.2519	-.0052	.6030	.1682	-.0043	-.0717	-.0058	.0132
3.3	.8531	.2515	-.0041	.6198			-.0722		
3.4	.8782	.2511	-.0032	.6365	.1676	-.0026	-.0726	-.0036	.0089
3.5	.9033	.2508	-.0024	.6533			-.0730		
3.6	.9284	.2506	-.0019	.6700	.1672	-.0015	-.0732	-.0021	.0057
3.7	.9534	.2504	-.0014	.6867			-.0734		
3.8	.9785	.2503	-.0011	.7034	.1669	-.0010	-.0735	-.0012	.0036
3.9	1.0035	.2502	-.0008	.7201			-.0736		
4.0	1.0285	.2502	-.0006	.7368	.1668	-.0004	-.0737	-.0006	.0022
4.1	1.0535	.2501	-.0004	.7535			-.0738		
4.2	1.0785	.2501	-.0003	.7701	.1667	-.0001	-.0738	-.0003	.0012
4.3		.2500	-.0002	.7868			-.0738		
4.4			-.0001	.8035		-.0001	-.0738	-.0001	.0007
4.5			-.0000	.8201			-.0739		
4.6				.8368		-.0000		-.0000	.0003
4.7									
4.8									.0001
4.9									
5.0									.0000

TABLE 2. TWO-DIMENSIONAL SYMMETRICAL

TRANSITION BOUNDARY LAYER

η	$1 - F_0$	$-F_0'$	F_2	F_2'	G_4	G_4'	H_4	H_4'
0.0	0	0.4959	0	-0.1119	0	-0.0977	0	0.0318
.2	.0991	.4953	-.0224	-.1113	-.0195	-.0970	.0064	.0320
.4	.1979	.4917	-.0443	-.1077	-.0386	-.0932	.0129	.0338
.6	.2954	.4825	-.0651	-.0988	-.0565	-.0846	.0200	.0378
.8	.3904	.4660	-.0835	-.0841	-.0721	-.0704	.0281	.0433
1.0	.4813	.4416	-.0983	-.0638	-.0843	-.0517	.0374	.0490
1.2	.5666	.4095	-.1087	-.0397	-.0926	-.0301	.0476	.0529
1.4	.6447	.3708	-.1141	-.0138	-.0964	-.0078	.0583	.0530
1.6	.7146	.3275	-.1143	.0111	-.0958	.0131	.0685	.0480
1.8	.7755	.2818	-.1099	.0328	-.0913	.0307	.0771	.0376
2.0	.8273	.2360	-.1015	.0496	-.0838	.0440	.0832	.0228
2.2	.8701	.1924	-.0904	.0605	-.0741	.0521	.0861	.0054
2.4	.9045	.1526	-.0777	.0656	-.0633	.0554	.0854	-.0123
2.6	.9315	.1177	-.0645	.0654	-.0522	.0544	.0813	-.0281
2.8	.9520	.0883	-.0518	.0610	-.0417	.0502	.0744	-.0404
3.0	.9672	.0644	-.0403	.0540	-.0323	.0440	.0654	-.0481
3.2	.9781	.0457	-.0303	.0454	-.0242	.0368	.0555	-.0511
3.4	.9858	.0315	-.0221	.0366	-.0176	.0295	.0453	-.0500
3.6	.9910	.0211	-.0156	.0283	-.0124	.0227	.0356	-.0458
3.8	.9944	.0138	-.0107	.0211	-.0084	.0168	.0271	-.0396
4.0	.9966	.0088	-.0071	.0151	-.0056	.0119	.0199	-.0325
4.2	.9980	.0054	-.0046	.0104	-.0036	.0082	.0141	-.0254
4.4	.9989	.0032	-.0029	.0070	-.0022	.0055	.0097	-.0191
4.6	.9994	.0019	-.0018	.0045	-.0014	.0035	.0064	-.0137
4.8	.9996	.0011	-.0010	.0028	-.0008	.0021	.0041	-.0095
5.0	.9998	.0006	-.0006	.0017	-.0005	.0013	.0026	-.0063
5.2	.9999	.0003	-.0003	.0010	-.0003	.0008	.0015	-.0040
5.4	1.0000	.0002	-.0002	.0006	-.0002	.0005	.0009	-.0025
5.6		.0001	-.0001	.0003	-.0001	.0003	.0005	-.0015
5.8		.0000	-.0000	.0002	-.0000	.0002	.0003	-.0009
6.0				.0001		.0000	.0001	-.0005
6.2				.0000			.0001	-.0002
6.4							.0000	-.0001
6.6								-.0001
6.8								-.0000

TABLE 3. ROTATIONALLY SYMMETRICAL FLOW

BOUNDARY LAYER

η	f_1	f_1'	f_1''	g_3	g_3'	g_3''	h_3	h_3'	h_3''
0.0	0	0	0.9277	0	0	1.0475	0	0	0.0448
.1	.0046	.0903	.8777	.0051	.0998	.9477	.0002	.0044	.0448
.2	.0179	.1755	.8277	.0196	.1896	.8488	.0009	.0090	.0444
.3	.0395	.2558	.7778	.0427	.2696	.7517	.0020	.0133	.0434
.4	.0689	.3311	.7282	.0732	.3400	.6574	.0036	.0176	.0416
.5	.1056	.4014	.6788	.1104	.4012	.5666	.0055	.0217	.0391
.6	.1490	.4669	.6300	.1532	.4535	.4802	.0079	.0254	.0356
.7	.1988	.5275	.5819	.2008	.4974	.3986	.0106	.0288	.0314
.8	.2544	.5833	.5348	.2524	.5334	.3227	.0136	.0316	.0265
.9	.3153	.6345	.4888	.3072	.5621	.2528	.0169	.0340	.0210
1.0	.3811	.6811	.4443	.3646	.5842	.1895	.0204	.0358	.0152
1.1	.4514	.7234	.4014	.4239	.6002	.1328	.0241	.0370	.0091
1.2	.5256	.7614	.3604	.4845	.6110	.0832	.0278	.0377	.0032
1.3	.6035	.7954	.3215	.5459	.6171	.0403	.0316	.0377	-.0026
1.4	.6846	.8258	.2850	.6078	.6193	.0044	.0353	.0372	-.0080
1.5	.7686	.8526	.2508	.6696	.6182	-.0251	.0390	.0361	-.0127
1.6	.8550	.8761	.2192	.7313	.6144	-.0483	.0425	.0346	-.0168
1.7	.9437	.8966	.1901	.7925	.6087	-.0657	.0459	.0327	-.0202
1.8	1.0342	.9142	.1637	.8530	.6015	-.0780	.0491	.0306	-.0228
1.9	1.1264	.9294	.1398	.9127	.5932	-.0857	.0520	.0282	-.0244
2.0	1.2200	.9422	.1185	.9716	.5845	-.0894	.0547	.0258	-.0254
2.1	1.3148	.9530	.0996	1.0296	.5755	-.0898	.0572	.0233	-.0256
2.2	1.4106	.9622	.0831	1.0867	.5666	-.0876	.0594	.0207	-.0252
2.3	1.5072	.9698	.0688	1.1430	.5580	-.0834	.0613	.0182	-.0243
2.4	1.6045	.9760	.0564	1.1984	.5500	-.0776	.0630	.0158	-.0229
2.5	1.7024	.9811	.0458	1.2530	.5425	-.0709	.0645	.0136	-.0212
2.6	1.8007	.9853	.0370	1.3069	.5358	-.0637	.0657	.0116	-.0193
2.7	1.8994	.9886	.0296	1.3602	.5298	-.0563	.0668	.0097	-.0174
2.8	1.9984	.9912	.0234	1.4129	.5245	-.0490	.0677	.0081	-.0153
2.9	2.0977	.9932	.0184	1.4651	.5200	-.0420	.0684	.0067	-.0133
3.0	2.1971	.9949	.0143	1.5169	.5161	-.0356	.0690	.0054	-.0114
3.1	2.2966	.9962	.0110	1.5683	.5128	-.0297	.0695	.0044	-.0097
3.2	2.3963	.9972	.0085	1.6195	.5102	-.0245	.0699	.0035	-.0082
3.3	2.4961	.9979	.0064	1.6704	.5079	-.0200	.0702	.0028	-.0067
3.4	2.5959	.9985	.0048	1.7211	.5061	-.0161	.0705	.0022	-.0054
3.5	2.6958	.9989	.0036	1.7716	.5047	-.0128	.0706	.0016	-.0044
3.6	2.7957	.9992	.0026	1.8220	.5036	-.0101	.0708	.0013	-.0035
3.7	2.8956	.9995	.0020	1.8723	.5027	-.0078	.0709	.0010	-.0028
3.8	2.9956	.9996	.0014	1.9226	.5020	-.0060	.0710	.0007	-.0021
3.9	3.0955	.9997	.0010	1.9727	.5015	-.0046	.0711	.0006	-.0016
4.0	3.1955	.9998	.0007	2.0229	.5011	-.0034	.0711	.0004	-.0013
4.1	3.2955	.9999	.0005	2.0730	.5008	-.0026	.0711	.0002	-.0009
4.2	3.3955	.9999	.0004	2.1230	.5006	-.0019	.0712	.0002	-.0007
4.3	3.4955	.9999	.0003	2.1731	.5004	-.0014	.0712	.0002	-.0006
4.4	3.5954	.9999	.0002	2.2231	.5003	-.0010	.0712	.0002	-.0004
4.5	3.6954	1.0000	.0001	2.2731	.5002	-.0007		.0000	-.0002
4.6			.0001	2.3231	.5001	-.0005			-.0002
4.7			.0000	2.3732	.5001	-.0004			-.0001
4.8					.5000	-.0002			-.0001
4.9						-.0001			-.0000
5.0						-.0001			
5.1						-.0001			
5.2						-.0000			

TABLE 4. ROTATIONALLY SYMMETRICAL

FLOW BOUNDARY

η	g_5	g_5'	g_5''	h_5	h_5'	h_5''	k_5	k_5'	k_5''	j_5	j_5'	j_5''	q_5	q_5'	q_5''
0.0	0	0	0.9054	0	0	0.0506	0	0	0.1768	0	0	0.0291	0	0	-0.0244
.2	.0168	.1612	.7075	.0010	.0101	.0500	.0029	.0255	.0790	.0006	.0058	.0278	-.0005	-.0049	-.0242
.4	.0619	.2838	.5210	.0040	.0198	.0467	.0090	.0324	-.0068	.0022	.0107	.0210	-.0019	-.0096	-.0230
.6	.1279	.3709	.3541	.0089	.0285	.0396	.0148	.0241	-.0724	.0047	.0137	.0074	-.0043	-.0140	-.0203
.8	.2082	.4270	.2123	.0153	.0354	.0289	.0179	.0051	-.1132	.0075	.0134	-.0104	-.0075	-.0176	-.0164
1.0	.2971	.4576	.0984	.0229	.0399	.0159	.0165	-.0195	-.1284	.0099	.0096	-.0280	-.0113	-.0204	-.0115
1.2	.3899	.4683	.0128	.0311	.0417	.0024	.0101	-.0447	-.1204	.0111	.0025	-.0412	-.0156	-.0222	-.0062
1.4	.4834	.4645	-.0459	.0394	.0409	-.0099	-.0011	-.0665	-.0948	.0108	-.0064	-.0467	-.0201	-.0229	-.0008
1.6	.5751	.4515	-.0808	.0473	.0379	-.0195	-.0161	-.0819	-.0585	.0085	-.0156	-.0437	-.0247	-.0226	.0043
1.8	.6637	.4335	-.0964	.0544	.0334	-.0256	-.0334	-.0897	-.0194	.0046	-.0234	-.0336	-.0291	-.0212	.0085
2.0	.7484	.4139	-.0974	.0606	.0279	-.0282	-.0515	-.0899	.0161	-.0006	-.0287	-.0191	-.0331	-.0192	.0117
2.2	.8293	.3952	-.0888	.0656	.0223	-.0277	-.0689	-.0838	.0432	-.0067	-.0310	-.0037	-.0367	-.0167	.0136
2.4	.9066	.3787	-.0750	.0695	.0170	-.0250	-.0847	-.0733	.0599	-.0129	-.0304	.0095	-.0398	-.0139	.0142
2.6	.9810	.3652	-.0594	.0724	.0124	-.0209	-.0981	-.0605	.0662	-.0187	-.0275	.0187	-.0423	-.0111	.0136
2.8	1.0530	.3549	-.0445	.0745	.0086	-.0165	-.1089	-.0474	.0642	-.0238	-.0232	.0234	-.0442	-.0085	.0122
3.0	1.1231	.3473	-.0317	.0760	.0058	-.0122	-.1171	-.0352	.0565	-.0279	-.0184	.0241	-.0457	-.0062	.0103
3.2	1.1920	.3420	-.0215	.0769	.0037	-.0086	-.1231	-.0249	.0460	-.0311	-.0137	.0219	-.0467	-.0044	.0082
3.4	1.2601	.3385	-.0139	.0775	.0023	-.0058	-.1272	-.0168	.0350	-.0334	-.0097	.0181	-.0474	-.0029	.0061
3.6	1.3275	.3363	-.0086	.0778	.0013	-.0037	-.1300	-.0109	.0251	-.0351	-.0065	.0138	-.0479	-.0019	.0044
3.8	1.3947	.3350	-.0051	.0780	.0008	-.0022	-.1317	-.0067	.0170	-.0361	-.0042	.0099	-.0482	-.0012	.0030
4.0	1.4616	.3342	-.0029	.0781	.0004	-.0013	-.1328	-.0040	.0109	-.0368	-.0025	.0067	-.0484	-.0007	.0019
4.2	1.5284	.3338	-.0016	.0782	.0002	-.0007	-.1334	-.0022	.0066	-.0372	-.0015	.0042	-.0485	-.0004	.0012
4.4	1.5951	.3336	-.0008	.0782	.0001	-.0004	-.1337	-.0012	.0039	-.0374	-.0008	.0026	-.0486	-.0002	.0007
4.6	1.6618	.3334	-.0004	.0782	.0000	-.0002	-.1339	-.0006	.0022	-.0375	-.0004	.0014	-.0486	-.0001	.0004
4.8	1.7285	.3334	-.0002	.0783	.0000	-.0001	-.1340	-.0003	.0012	-.0376	-.0002	.0008	-.0486	-.0000	.0002
5.0	1.7952	.3334	-.0001	.0783	.0000	-.0000	-.1340	-.0001	.0006	-.0376	-.0001	.0004	-.0486	-.0000	.0001
5.2	1.8618	.3334	-.0000	.0783	.0000	-.0000	-.1340	-.0000	.0003	-.0376	-.0000	.0002	-.0486	-.0000	.0000
5.4	1.9285	.3333	-.0000	.0783	.0000	-.0000	-.1340	-.0000	.0001	-.0376	-.0000	.0001			
5.6									.0000			.0000			

TABLE 5. ROTATIONALLY SYMMETRICAL

TRANSFER BOUNDARY LAYER

σ	$-(F_0')_0$
0.5	0.4129
.7	.4705
1	.5390
1/0.395	.7599
10	1.2389
100	2.7365

TABLE 6. TWO-DIMENSIONAL SYMMETRICAL TRANSFER

BOUNDARY LAYER. σ LARGE

ξ	$1 - \Phi_0$	$-\Phi_0'$	Φ_2	Φ_2'
0.0	0	1.1198	0	-0.4799
.1	.1120	1.1187	-.0479	-.4780
.2	.2235	1.1109	-.0952	-.4647
.3	.3337	1.0900	-.1401	-.4293
.4	.4409	1.0504	-.1801	-.3637
.5	.5430	.9883	-.2118	-.2647
.6	.6378	.9023	-.2320	-.1361
.7	.7228	.7947	-.2384	.0099
.8	.7962	.6711	-.2301	.1541
.9	.8567	.5402	-.2083	.2747
1.0	.9043	.4120	-.1765	.3530
1.1	.9396	.2959	-.1395	.3794
1.2	.9641	.1989	-.1023	.3565
1.3	.9801	.1245	-.0693	.2981
1.4	.9897	.0720	-.0432	.2232
1.5	.9951	.0383	-.0246	.1500
1.6	.9979	.0186	-.0128	.0904
1.7	.9992	.0082	-.0060	.0488
1.8	.9997	.0033	-.0026	.0236
1.9	.9999	.0012	-.0010	.0102
2.0	1.0000	.0004	-.0003	.0040
2.1		.0001	-.0001	.0014
2.2		.0000	-.0000	.0004
2.3				.0001
2.4				.0000